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NEW CONGRUENCES INVOLVING PRODUCTS OF TWO BINOMIAL COEFFICIENTS

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ABSTRACT. Let $p > 3$ be a prime and let a be a positive integer. We show that if $p \equiv 1 \pmod{4}$ or $a > 1$ then

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p^a} \right) \pmod{p^3}$$

with $(-)$ the Jacobi symbol, which confirms a conjecture of Z.-W. Sun. We also establish the following new congruences:

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} &\equiv \left(\frac{p}{3} \right) \frac{2^p + 1}{3} \pmod{p^2}, \\ \sum_{k=0}^{(p-1)/2} \frac{\binom{6k}{3k} \binom{3k}{k}}{(2k+1)432^k} &\equiv \left(\frac{p}{3} \right) \frac{3^p + 1}{4} \pmod{p^2}, \\ \sum_{k=0}^{(p-1)/2} \frac{\binom{4k}{2k} \binom{2k}{k}}{(2k+1)64^k} &\equiv \left(\frac{-1}{p} \right) 2^{p-1} \pmod{p^2}. \end{aligned}$$

Note that in 2003 Rodriguez-Villeguez posed conjectures on

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k}, \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k}, \sum_{k=1}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k}, \sum_{k=1}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k}$$

modulo p^2 which were later proved.

Key words and phrases. Central binomial coefficients, congruences.

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1. INTRODUCTION

Let $p > 3$ be a prime. In 2003, via his analysis of the p -adic analogues of Gaussian hypergeometric series and the Calabi- Yau manifolds, Rodriguez-Villegas [RV] conjectured the following congruences:

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} &\equiv \left(\frac{-1}{p}\right) \pmod{p^2}, & \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} &\equiv \left(\frac{p}{3}\right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} &\equiv \left(\frac{-2}{p}\right) \pmod{p^2}, & \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} &\equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \end{aligned}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Jacobi symbol. They were soon proved by E. Mortenson [M1, M2] via the Gross-Koblitz formula and the p -adic Γ -function. Note that

$$\begin{aligned} \left(\frac{-1/2}{k}\right) &= \frac{\binom{2k}{k}^2}{16^k}, & \left(\frac{-1/3}{k}\right) \left(\frac{-2/3}{k}\right) &= \frac{\binom{2k}{k} \binom{3k}{k}}{27^k}, \\ \left(\frac{-1/4}{k}\right) \left(\frac{-3/4}{k}\right) &= \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k}, & \left(\frac{-1/6}{k}\right) \left(\frac{-5/6}{k}\right) &= \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \end{aligned}$$

for all $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. In 2011 Z. W. Sun [Su11] showed further that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) + p^2 E_{p-3} \pmod{p^3}, \quad (1.1)$$

where E_0, E_1, E_2, \dots are the Euler numbers given by

$$E_0 = 1, \text{ and } E_n = - \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} E_{n-2k} \quad (n = 1, 2, 3, \dots).$$

He also conjectured that

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)27^k} &\equiv \left(\frac{p}{3}\right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{(2k+1)64^k} &\equiv \left(\frac{-1}{p}\right) - 3p^2 E_{p-3} \pmod{p^3}, \\ \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{(2k+1)432^k} &\equiv \left(\frac{p}{3}\right) \pmod{p^2},\end{aligned}$$

which were confirmed by Z.-H. Sun [S16]. Note that Z.-W. Sun [Su14] determined

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)16^k} \quad \text{and} \quad \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2}{(2k+1)16^k}$$

modulo p^3 .

In this paper we first establish the following result.

Theorem 1.1. *Let p be any odd prime.*

(i) *We have*

$$\sum_{k=0}^{\lfloor 3p/4 \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv \begin{cases} 1 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ -1 + p^2 / (2^{\binom{(p-3)/2}{(p-3)/4}}) \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \quad (1.2)$$

(ii) *For each $a = 2, 3, 4, \dots$, we have*

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p^a}\right) \pmod{p^3}. \quad (1.3)$$

Remark 1.1. . Part (i) in the case $p \equiv 1 \pmod{4}$ and part (ii) were conjectured by Sun [Su11].

Our second theorem is as follows.

Theorem 1.2. *Let $p > 3$ be a prime. Then we have*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \frac{2^p + 1}{3} \pmod{p^2}, \quad (1.4)$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{6k}{3k} \binom{3k}{k}}{(2k+1)432^k} \equiv \left(\frac{p}{3}\right) \frac{3^p + 1}{4} \pmod{p^2}, \quad (1.5)$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{4k}{2k} \binom{2k}{k}}{(2k+1)64^k} \equiv \left(\frac{-1}{p}\right) 2^{p-1} \pmod{p^2}. \quad (1.6)$$

Remark 1.2. We are also able to show the congruence

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)27^k} \equiv \left(\frac{p}{3}\right) (3^p + 2 - 2^{p+1}) \pmod{p^2}$$

for any prime $p > 3$.

2. PROOF OF THEOREM 1.1

Lemma 2.1. (Sun [Su11, (1.4)]) *For any prime $p > 3$ we have*

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} 4E_{p-3} \pmod{p}. \quad (2.1)$$

Proof of Theorem 1.1(i). In view of (1.1), (1.2) has the following equation form:

$$\sum_{k=(p+1)/2}^{\lfloor 3p/4 \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv -p^2 E_{p-3} + \frac{1 - (-1)^{(p-1)/2}}{2} \cdot \frac{p^2}{2^{\binom{(p-3)/2}{\lfloor p/4 \rfloor}^2}} \pmod{p^3}. \quad (2.2)$$

By [Su11, Lemma 2.1],

$$k \binom{2k}{k} \binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2} \text{ for all } k = 1, \dots, p-1.$$

Thus

$$\begin{aligned} \sum_{k=(p+1)/2}^{\lfloor 3p/4 \rfloor} \frac{\binom{2k}{k}^2}{16^k} &\equiv \sum_{k=(p+1)/2}^{\lfloor 3p/4 \rfloor} \frac{4p^2}{k^2 \binom{2(p-k)}{p-k}^2 16^k} = \sum_{j=\lfloor p/4 \rfloor + 1}^{(p-1)/2} \frac{4p^2}{(p-j)^2 \binom{2j}{j}^2 16^{p-j}} \\ &\equiv \frac{p^2}{4} \sum_{j=\lfloor p/4 \rfloor + 1}^{(p-1)/2} \frac{16^j}{j^2 \binom{2j}{j}^2} \pmod{p^3} \end{aligned}$$

and hence we have reduced (2.2) to the following simpler form

$$\sum_{k=\lfloor n/2 \rfloor + 1}^n \frac{16^k}{k^2 \binom{2k}{k}^2} \equiv -4E_{p-3} + \frac{1 - (-1)^n}{\binom{n-1}{\lfloor n/2 \rfloor}^2} \pmod{p}, \quad (2.3)$$

where $n = (p-1)/2$.

For each $k = 0, \dots, n$, clearly

$$\binom{n}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod{p}.$$

Thus

$$\sum_{k=\lfloor n/2 \rfloor + 1}^n \frac{16^k}{k^2 \binom{2k}{k}^2} \equiv \sum_{k=\lfloor n/2 \rfloor + 1}^n \frac{1}{k^2 \binom{n}{k}^2} \equiv 4 \sum_{k=\lfloor n/2 \rfloor + 1}^n \frac{1}{\binom{n-1}{k-1}^2} \pmod{p}.$$

Note that

$$\sum_{k=\lfloor n/2 \rfloor + 1}^n \frac{1}{\binom{n-1}{k-1}^2} = \sum_{k=\lfloor n/2 \rfloor}^{n-1} \frac{1}{\binom{n-1}{k}^2} = \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}^2} + \frac{1 - (-1)^n}{4 \binom{n-1}{\lfloor n/2 \rfloor}^2}$$

and

$$\sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}^2} = \frac{2n^2}{n+1} \sum_{k=1}^n \frac{1}{k \binom{2n+1-k}{n-k}} \quad (2.4)$$

(cf. [SWZ]). So we have

$$\begin{aligned} & \sum_{k=\lfloor n/2 \rfloor + 1}^n \frac{16^k}{k^2 \binom{2k}{k}^2} - \frac{1 - (-1)^n}{\binom{n-1}{\lfloor n/2 \rfloor}^2} \\ & \equiv \frac{4n^2}{n+1} \sum_{k=1}^n \frac{1}{k \binom{2n+1-k}{n-k}} \equiv 2 \sum_{k=1}^n \frac{1}{k \binom{-k}{n-k}} \pmod{p}. \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k \binom{-k}{n-k}} &= \sum_{k=1}^n \frac{(-1)^{n-k}}{k \binom{n-1}{k-1}} = n \sum_{k=1}^n \frac{(-1)^{n-k}}{k^2 \binom{n}{k}} \\ &\equiv \frac{(-1)^{n-1}}{2} \sum_{k=1}^n \frac{4^k}{k^2 \binom{2k}{k}} \pmod{p}. \end{aligned}$$

Therefore, with the help of Lemma 2.1, we finally obtain

$$\sum_{k=\lfloor n/2 \rfloor + 1}^n \frac{16^k}{k^2 \binom{2k}{k}^2} - \frac{1 - (-1)^n}{\binom{n-1}{\lfloor n/2 \rfloor}^2} \equiv (-1)^{n-1} \sum_{k=1}^n \frac{4^k}{k^2 \binom{2k}{k}} \equiv -4E_{p-3} \pmod{p}.$$

This proves (2.3) and hence (1.2) follows. \square

Now we give a lemma which is a natural extension of (1.1).

Lemma 2.2. *Let $p > 3$ be a prime and let a be any positive integer. Then*

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p^a} \right) + \left(\frac{-1}{p^{a-1}} \right) p^2 E_{p-3} \pmod{p^3}. \quad (2.5)$$

Proof. Theorem 1.2 of Sun [Su13] states that for any $d = 0, \dots, (p-1)/2$ we have

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{2k}{k+d}}{16^k} \equiv \left(\frac{-1}{p} \right) + \frac{(-1)^d}{4} p^2 E_{p-3} \left(d + \frac{1}{2} \right) \pmod{p^3},$$

where $E_{p-3}(x)$ denotes the Euler polynomial of the degree $p-3$.

In the case $d = 0$ this yields (1.1). Modifying this proof of (1.1) slightly we immediately get (2.5). \square

In 1852, Kummer proved that for any $m, n \in \mathbb{N}$ the p -adic valuation of the binomial coefficient $\binom{m+n}{m}$ is equal to the number of *carry-overs* when performing the addition of m and n written in base p .

Lemma 2.3. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. For any $k = 1, 2, \dots, (p^a - 1)/2$, we have*

$$\text{ord}_p \left(\binom{p^a - k}{\frac{p^a - 1}{2} - k} \right) \leq a - 1.$$

Proof. It is well known that

$$\text{ord}_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor.$$

Thus

$$\text{ord}_p \left(\binom{p^a - k}{\frac{p^a - 1}{2} - k} \right) = \sum_{j=1}^{a-1} \left(\left\lfloor \frac{p^a - k}{p^j} \right\rfloor - \left\lfloor \frac{(p^a + 1)/2}{p^j} \right\rfloor - \left\lfloor \frac{(p^a - 1)/2 - k}{p^j} \right\rfloor \right)$$

does not exceed $a - 1$ as each term in the sum is at most one. This concludes the proof. \square

Proof of Theorem 1.1(ii). In view of Lemma 2.2, we just need to verify that

$$\sum_{k=(p^a+1)/2}^{\lfloor 3p^a/4 \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p^{a-1}} \right) p^2 E_{p-3} \pmod{p^3}. \quad (2.6)$$

Let k and l be positive integers with $k + l = p^a$ and $0 < l < p^a/2$. Then

$$\frac{\binom{2k}{k}^2}{\binom{2p^a-2}{p^a-1}^2} = \frac{(2p^a - 2l)!^2}{(2p^a - 2)!^2} \left(\frac{(p^a - 1)!}{(p^a - l)!} \right)^4 = \frac{\prod_{0 < i < l} (p^a - i)^4}{\prod_{1 < j < 2l} (2p^a - j)^2}$$

and hence

$$\frac{\binom{2k}{k}^2}{\binom{2p^a-2}{p^a-1}^2} \cdot \frac{(2l - 1)!^2}{(l - 1)!^4} = \frac{\prod_{0 < i < l} (1 - p^a/i)^4}{\prod_{1 < j < 2l} (1 - 2p^a/j)^2} \equiv 1 \pmod{p}.$$

Note that

$$\binom{2p^a - 2}{p^a - 1}^2 = p^{2a} \prod_{j=2}^{p^a-1} \binom{2p^a - j}{j}^2 \equiv p^{2a} \pmod{p^{2a+1}}$$

and

$$\binom{2k}{k}^2 = \binom{p^a + (2k - p^a)}{0p^a + k}^2 \equiv \binom{2k - p^a}{k}^2 = 0 \pmod{p^2}$$

by Lucas' theorem. So we have

$$\frac{l^2}{4} \binom{2l}{l}^2 = \frac{(2l-1)!^2}{(l-1)!^4} \not\equiv 0 \pmod{p^{2a}}$$

and

$$\binom{2k}{k}^2 \equiv p^{2a} \frac{(l-1)!^4}{(2l-1)!^2} = \frac{4p^{2a}}{l^2 \binom{2l}{l}^2} \pmod{p^3}.$$

Therefore

$$\begin{aligned} \sum_{k=(p^a+1)/2}^{\lfloor 3p^a/4 \rfloor} \frac{\binom{2k}{k}^2}{16^k} &\equiv \sum_{k=(p^a+1)/2}^{\lfloor 3p^a/4 \rfloor} \frac{4p^{2a}}{16^k (p^a - k)^2 \binom{2(p^a-k)}{p^a-k}^2} \\ &\equiv \frac{p^{2a}}{4} \sum_{l=\lfloor p^a/4 \rfloor + 1}^{(p^a-1)/2} \frac{16^l}{l^2 \binom{2l}{l}^2} \pmod{p^3}. \end{aligned}$$

For $k = 1, \dots, (p^a - 1)/2$, clearly

$$\begin{aligned} \frac{\binom{(p^a-1)/2}{k}}{\binom{2k}{k}/(-4)^k} &= \frac{\binom{(p^a-1)/2}{k}}{\binom{-1/2}{k}} = \prod_{j=0}^{k-1} \frac{(p^a - 1)/2 - j}{-1/2 - j} \\ &= \prod_{j=0}^{k-1} \left(1 - \frac{p^a}{2j+1}\right) \equiv 1 \pmod{p}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{k=(p^a+1)/2}^{\lfloor 3p^a/4 \rfloor} \frac{\binom{2k}{k}^2}{16^k} &\equiv \frac{p^{2a}}{4} \sum_{k=\lfloor p^a/4 \rfloor + 1}^{(p^a-1)/2} \frac{1}{k^2 \binom{(p^a-1)/2}{k}^2} \\ &\equiv p^{2a} \sum_{k=\lfloor p^a/4 \rfloor + 1}^{(p^a-1)/2} \frac{1}{\binom{(p^a-3)/2}{k-1}^2} \pmod{p^3}. \end{aligned}$$

So (2.6) is reduced to

$$p^{2a-2} \sum_{k=\lfloor p^a/4 \rfloor}^{(p^a-3)/2} \frac{1}{\binom{(p^a-3)/2}{k}^2} \equiv - \left(\frac{-1}{p^{a-1}} \right) E_{p-3} \pmod{p}. \quad (2.7)$$

If $p^a \equiv 1 \pmod{4}$, then $(p^a - 3)/2$ is odd and hence

$$\sum_{k=\lfloor p^a/4 \rfloor}^{(p^a-3)/2} \frac{1}{\binom{(p^a-3)/2}{k}^2} = \frac{1}{2} \sum_{k=0}^{(p^a-3)/2} \frac{1}{\binom{(p^a-3)/2}{k}^2}.$$

If $p^a \equiv 3 \pmod{4}$, then $a \in \{3, 5, \dots\}$ and

$$\sum_{k=\lfloor p^a/4 \rfloor}^{(p^a-3)/2} \frac{1}{\binom{(p^a-3)/2}{k}^2} = \frac{1}{2} \sum_{k=0}^{(p^a-3)/2} \frac{1}{\binom{(p^a-3)/2}{k}^2} + \frac{1}{2} \cdot \frac{1}{\binom{(p^a-3)/2}{(p^a-3)/4}^2}.$$

In the case $p^a \equiv 3 \pmod{4}$, as the fractional parts of $(p^a - 3)/(2p)$ and $(p^a - 3)/(4p)$ are $(p - 3)/(2p)$ and $(p - 3)/(4p)$ respectively, we have

$$\left\lfloor \frac{(p^a - 3)/2}{p} \right\rfloor = 2 \left\lfloor \frac{(p^a - 3)/4}{p} \right\rfloor$$

and hence

$$\text{ord}_p \left(\frac{\binom{(p^a - 3)/2}{(p^a - 3)/4}}{\binom{(p^a - 3)/2}{(p^a - 3)/4}} \right)^2 = 2 \sum_{j=1}^{a-1} \left(\left\lfloor \frac{(p^a - 3)/2}{p^j} \right\rfloor - 2 \left\lfloor \frac{(p^a - 3)/4}{p^j} \right\rfloor \right) < 2a - 2.$$

No matter $p^a \equiv 1 \pmod{4}$ or not, we always have

$$p^{2a-2} \sum_{k=\lfloor p^a/4 \rfloor}^{(p^a-3)/2} \frac{1}{\binom{(p^a-3)/2}{k}^2} \equiv \frac{p^{2a-2}}{2} \sum_{k=0}^{(p^a-3)/2} \frac{1}{\binom{(p^a-3)/2}{k}^2} \pmod{p}.$$

So (2.7) has the following equivalent form:

$$p^{2a-2} \sum_{k=0}^{(p^a-3)/2} \frac{1}{\binom{(p^a-3)/2}{k}^2} \equiv -2 \left(\frac{-1}{p^{a-1}} \right) E_{p-3} \pmod{p}. \quad (2.8)$$

The identity (2.4) with $n = (p^a - 1)/2$ yields that

$$\sum_{k=0}^{(p^a-3)/2} \frac{1}{\binom{(p^a-3)/2}{k}^2} = \frac{2((p^a - 1)/2)^2}{(p^a + 1)/2} \sum_{k=1}^{(p^a-1)/2} \frac{1}{k \binom{p^a-k}{(p^a-1)/2-k}}.$$

So (2.8) is reduced to

$$p^{2a-2} \sum_{k=1}^{(p^a-1)/2} \frac{1}{k \binom{p^a-k}{(p^a+1)/2}} \equiv -2 \left(\frac{-1}{p^{a-1}} \right) E_{p-3} \pmod{p}. \quad (2.9)$$

In view of Lemma 2.3, if $1 \leq k \leq (p^a - 1)/2$ and $p^{a-1} \nmid k$, then

$$\frac{p^{2a-2}}{k \binom{p^a-k}{(p^a+1)/2}} \equiv 0 \pmod{p}.$$

Thus

$$\begin{aligned} p^{2a-2} \sum_{k=1}^{(p^a-1)/2} \frac{1}{k \binom{p^a-k}{(p^a+1)/2}} &\equiv p^{2a-2} \sum_{j=1}^{(p-1)/2} \frac{1}{p^{a-1} j \binom{p^a-p^{a-1}j}{(p^a+1)/2}} \\ &= \frac{p^a+1}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j(p-j) \binom{p^a-p^{a-1}j-1}{(p^a-1)/2}} \\ &\equiv -\frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j^2 \binom{p^a-p^{a-1}j-1}{(p^a-1)/2}} \pmod{p}. \end{aligned}$$

For each $j = 1, \dots, (p-1)/2$, by Lucas' theorem we have

$$\begin{aligned} \binom{p^{a-1}(p-j)-1}{(p^a-1)/2} &= \binom{p^{a-1}(p-1-j)+p^{a-1}-1}{p^{a-1}(p-1)/2+(p^{a-1}-1)/2} \\ &\equiv \binom{p-1-j}{(p-1)/2} \binom{p^{a-1}-1}{(p^a-1)/2} \\ &\equiv (-1)^{(p^{a-1}-1)/2} \binom{p-j-1}{(p-1)/2} \pmod{p}, \end{aligned}$$

also

$$\begin{aligned} \binom{p-j-1}{(p-1)/2} &= \binom{p-1-j}{(p-1)/2-j} = (-1)^{(p-1)/2-j} \binom{-p+(p-1)/2}{(p-1)/2-j} \\ &\equiv (-1)^{(p-1)/2-j} \binom{(p-1)/2}{j} \equiv (-1)^{(p-1)/2-j} \binom{-1/2}{j} \\ &= (-1)^{(p-1)/2} \frac{\binom{2j}{j}}{4^j} \pmod{p}. \end{aligned}$$

Therefore

$$\begin{aligned} p^{2a-2} \sum_{k=1}^{(p^a-1)/2} \frac{1}{k \binom{p^a-k}{(p^a+1)/2}} &\equiv \frac{(-1)^{(p^{a-1}+1)/2}}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j^2 \binom{p-j-1}{(p-1)/2}} \\ &\equiv \frac{(-1)^{(p^{a-1}+1)/2}}{2} (-1)^{(p-1)/2} \sum_{j=1}^{(p-1)/2} \frac{4^j}{j^2 \binom{2j}{j}} \pmod{p}. \end{aligned}$$

This, together with (2.1), yields the desired (2.9).

The proof of Theorem 1.1(ii) is now complete. \square

3. PROOF OF THEOREM 1.2

Lemma 3.1. *Let $p > 3$ be a prime, and $m \in \{1, 2, \dots, (p-1)/2\}$. For any p -adic integer t , we have*

$$\binom{m+pt-1}{(p-1)/2} \binom{-1-pt-m}{(p-1)/2} \equiv \frac{pt}{m} \pmod{p^2}. \quad (3.1)$$

Proof. Since

$$\begin{aligned} \binom{m+pt-1}{(p-1)/2} &= \frac{\prod_{r=0}^{m-1} (pt+r) \times \prod_{s=1}^{(p-1)/2-m} (pt-s)}{((p-1)/2)!} \\ &\equiv \frac{(m-1)! pt (-1)^{(p-1)/2-m} ((p-1)/2-m)!}{((p-1)/2)!} \pmod{p^2}, \end{aligned}$$

and

$$\begin{aligned} \binom{-m-pt-1}{(p-1)/2} &= \frac{\prod_{j=1}^{(p-1)/2} (-m-pt-j)}{((p-1)/2)!} \\ &\equiv \frac{(-1)^{(p-1)/2} (m+1)(m+2) \cdots (m+(p-1)/2)}{((p-1)/2)!} \pmod{p}, \end{aligned}$$

we have

$$\begin{aligned} &\binom{m+pt-1}{(p-1)/2} \binom{-m-pt-1}{(p-1)/2} \\ &\equiv \frac{pt(m-1)! (-1)^m ((p-1)/2-m)! (m+1)(m+2) \cdots (m+(p-1)/2)}{((p-1)/2)!^2} \\ &= \frac{pt}{m} \frac{(-1)^m ((p-1)/2-m)! (m+(p-1)/2)!}{((p-1)/2)!^2} = \frac{pt}{m} (-1)^m \frac{\binom{p-1}{(p-1)/2}}{\binom{p-1}{(p-1)/2+m}} \\ &\equiv \frac{pt}{m} (-1)^m (-1)^{(p-1)/2} (-1)^{(p-1)/2+m} = \frac{pt}{m} \pmod{p^2}. \end{aligned}$$

This concludes the proof. \square

Remark 3.1. Let $p > 3$ be a prime and $m \in \{(p+1)/2, \dots, p-1\}$. For any p -adic integer t , by Lemma 3.1 we have

$$\begin{aligned} &\binom{m+pt-1}{(p-1)/2} \binom{-1-pt-m}{(p-1)/2} \\ &= \binom{(m-p)+p(t+1)-1}{(p-1)/2} \binom{-1-p(t+1)-(m-p)}{(p-1)/2} \\ &\equiv \frac{p(t+1)}{m-p} \equiv \frac{p(t+1)}{m} \pmod{p^2}. \end{aligned}$$

Lemma 3.2. *Let $p > 3$ be a prime. For $k \in \{1, 2, \dots, p-1\}$ and p -adic integer t , we have*

$$\binom{pt}{k} \binom{-1-pt}{k} \equiv -\frac{p^2 t^2}{k^2} - \frac{pt}{k} \pmod{p^3}. \quad (3.2)$$

Proof. This is almost trivial. In fact,

$$\begin{aligned} \binom{pt}{k} \binom{-1-pt}{k} &= \frac{pt}{pt-k} \binom{-1+pt}{k} \binom{-1-pt}{k} \\ &\equiv \frac{pt}{pt-k} \binom{-1}{k}^2 = \frac{pt(p^2 t^2 + ptk + k^2)}{(pt)^3 - k^3} \\ &\equiv -\frac{p^2 t^2}{k^2} - \frac{pt}{k} \pmod{p^3}. \end{aligned}$$

This proves (3.2). \square

Recall that those $H_n = \sum_{0 < k \leq n} 1/k$ with $n \in \mathbb{N}$ are called harmonic numbers. If a prime p does not divide an integer a , then we let $q_p(a)$ denote the Fermat quotient $(a^{p-1} - 1)/p$.

Lemma 3.3. (Lemma [L]). *For any prime $p > 3$, we have*

$$\begin{aligned} H_{\lfloor p/2 \rfloor} &\equiv -2q_p(2) \pmod{p}, \quad H_{\lfloor p/4 \rfloor} \equiv -3q_p(2) \pmod{p}, \\ H_{\lfloor p/3 \rfloor} &\equiv -\frac{3}{2}q_p(3) \pmod{p} \text{ and } H_{\lfloor p/6 \rfloor} \equiv -2q_p(2) - \frac{3}{2}q_p(3) \pmod{p}, \end{aligned}$$

where $q_p(2) = (2^{p-1} - 1)/p$ and $q_p(3) = (3^{p-1} - 1)/p$ stand for the Fermat quotients.

For $n \in \mathbb{N}$, define

$$S_n(x) = \sum_{k=0}^n \binom{x}{k} \binom{-1-x}{k} \quad \text{and} \quad T_n(x) = \sum_{k=0}^n \binom{x}{k} \binom{-1-x}{k} \frac{1+2x}{1+2k}.$$

By [S16, (2.2)] with $a = x+1$ and $b = 0$, we have

$$S_n(x) + S_n(x+1) = 2 \binom{x}{n} \binom{-2-x}{n}. \quad (3.3)$$

By [S16, (2.2)] with $b = 2$, we get

$$T_n(x) - T_n(x-1) = 2 \binom{x-1}{n} \binom{-x-1}{n}. \quad (3.4)$$

Proof of Theorem 1.2. For any p -adic integer a , we let $\langle a \rangle_p$ denote the least nonnegative integer r with $a \equiv r \pmod{p}$. For convenience, we also set $n = (p-1)/2$.

(i) For any p -adic integer $a \not\equiv 0 \pmod{p}$, by using (3.3) we get

$$\begin{aligned} & S_n(a) - (-1)^{\langle a \rangle_p} S_n(a - \langle a \rangle_p) \\ &= \sum_{k=0}^{\langle a \rangle_p - 1} (-1)^k (S_n(a - k) + S_n(a - k - 1)) \\ &= \sum_{k=0}^{\langle a \rangle_p - 1} (-1)^k 2 \binom{a - k - 1}{n} \binom{k - a - 1}{n} \end{aligned}$$

and hence

$$\begin{aligned} & S_n(a) - (-1)^{\langle a \rangle_p} S_n(pt) \\ &= 2 \sum_{k=0}^{\langle a \rangle_p - 1} (-1)^k \binom{\langle a \rangle_p + pt - k - 1}{n} \binom{-1 - pt - (\langle a \rangle_p - k)}{n}, \end{aligned}$$

where $t := (a - \langle a \rangle_p)/p$. By Lemma 3.2,

$$S_n(pt) = \sum_{k=0}^n \binom{pt}{k} \binom{-1 - pt}{k} \equiv 1 - \sum_{k=1}^n \frac{pt}{k} = 1 - ptH_n \pmod{p^2}.$$

So, with helps of Lemma 3.1 and Remark 3.1, we have

$$S_n(a) - (-1)^{\langle a \rangle_p} (1 - ptH_n) \equiv 2 \sum_{k=0}^{\langle a \rangle_p - 1} (-1)^k \frac{p(t + \delta_k)}{\langle a \rangle_p - k} \pmod{p^2}, \quad (3.5)$$

where δ_k takes 1 or 0 according as $\langle a \rangle_p - k > p/2$ or not.

Observe that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} = \sum_{k=0}^n \binom{-1/3}{k} \binom{-2/3}{k} = S_n(a)$$

with $a = -1/3$. Note that

$$\langle a \rangle_p = \begin{cases} (p-1)/3 & \text{if } p \equiv 1 \pmod{3}, \\ (2p-1)/3 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Hence

$$t := \frac{a - \langle a \rangle_p}{p} = \begin{cases} -1/3 & \text{if } p \equiv 1 \pmod{3}, \\ -2/3 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Case 1. $p \equiv 1 \pmod{3}$.

In this case, $\langle a \rangle_p = (p-1)/3$, $t = -1/3$, and $\delta_k = 0$ for all $k = 0, \dots, \langle a \rangle_p - 1$. So we have

$$\begin{aligned} & S_n \left(-\frac{1}{3} \right) - (-1)^{(p-1)/3} (1 - ptH_n) \\ & \equiv 2pt(-1)^{(p-1)/3} \sum_{j=1}^{(p-1)/3} \frac{(-1)^j}{j} = 2pt (H_{(p-1)/6} - H_{(p-1)/3}) \pmod{p^2}. \end{aligned}$$

Combining this with Lemma 3.3 and recalling that $t = -1/3$, we immediately obtain the desired congruence

$$S_n \left(-\frac{1}{3} \right) \equiv 1 + \frac{2}{3} p q_p(2) \pmod{p^2}.$$

Case 2. $p \equiv 2 \pmod{3}$.

In this case, we have $\langle a \rangle_p = (2p-1)/3$, $t = -2/3$ and

$$\delta_k = \begin{cases} 1 & \text{if } 0 \leq k < (p+1)/6, \\ 0 & \text{if } (p+1)/6 \leq k \leq \langle a \rangle_p - 1. \end{cases}$$

So we have

$$\begin{aligned} & S_n \left(-\frac{1}{3} \right) - (-1)^{(2p-1)/3} (1 - ptH_n) \\ & \equiv 2p(t+1) \sum_{k=0}^{(p-5)/6} \frac{(-1)^k}{\langle a \rangle_p - k} + 2pt \sum_{k=(p+1)/6}^{(2p-4)/3} \frac{(-1)^k}{\langle a \rangle_p - k} \\ & = 2p(t+1)(-1)^{(2p-1)/3} \sum_{j=(p+1)/2}^{(2p-1)/3} \frac{(-1)^j}{j} + 2pt(-1)^{(2p-1)/3} \sum_{j=1}^{(p-1)/2} \frac{(-1)^j}{j} \\ & = -2p(t+1) \sum_{j=1}^{(2p-1)/3} \frac{(-1)^j}{j} + 2p \sum_{j=1}^{(p-1)/2} \frac{(-1)^j}{j} \\ & = -2p(t+1) (H_{\lfloor p/3 \rfloor} - H_{\lfloor 2p/3 \rfloor}) + 2p (H_{\lfloor p/4 \rfloor} - H_{\lfloor p/2 \rfloor}) \pmod{p^2}. \end{aligned}$$

Note that

$$H_{\lfloor 2p/3 \rfloor} = \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k} + \frac{1}{p-k} \right) - \sum_{j=1}^{(p-1)/3} \frac{1}{p-j} \equiv H_{\lfloor p/3 \rfloor} \pmod{p}.$$

Therefore,

$$S_n \left(-\frac{1}{3} \right) + 1 - ptH_{\lfloor p/2 \rfloor} \equiv 2p (H_{\lfloor p/4 \rfloor} - H_{\lfloor p/2 \rfloor}) \pmod{p^2}.$$

This, together with Lemma 3.3 and the fact that $t = -2/3$, yields the desired congruence

$$S_n \left(-\frac{1}{3} \right) \equiv -1 - \frac{2}{3} p q_p(2) \pmod{p^2}.$$

In view of the above, we have completed the proof of (1.4).

(ii) For any p -adic integer a with $a(2a+1) \not\equiv 0 \pmod{p}$, if we set $t = (a - \langle a \rangle_p)/p$ then by (3.4) we have

$$\begin{aligned} T_n(a) - T_n(pt) &= \sum_{k=1}^{\langle a \rangle_p} (T_n(a - k + 1) - T_n(a - k)) \\ &= \sum_{k=1}^{\langle a \rangle_p} 2 \binom{a - k}{n} \binom{k - a - 2}{n} \\ &= 2 \sum_{k=1}^{\langle a \rangle_p} \binom{m_k + pt - 1}{n} \binom{-1 - pt - m_k}{n}, \end{aligned}$$

where $m_k = \langle a \rangle_p - k + 1$. In view of Lemmas 3.2 and 3.3,

$$\begin{aligned} T_n(pt) - (1 + 2pt) &= \sum_{k=0}^n \binom{pt}{k} \binom{-1 - pt}{k} \frac{1 + 2pt}{1 + 2k} - (1 + 2pt) \\ &\equiv \binom{pt}{n} \binom{-1 - pt}{n} \frac{1 + 2pt}{p} - \sum_{k=1}^{n-1} \frac{pt}{k(1 + 2k)} \\ &\equiv \left(-\frac{p^2 t^2}{n^2} - \frac{pt}{n} \right) \frac{1 + 2pt}{p} - \sum_{k=1}^{n-1} \frac{pt}{k(1 + 2k)} \\ &\equiv 2t + 2pt - pt \sum_{k=1}^{n-1} \frac{1}{k} + 2pt \sum_{k=1}^{n-1} \frac{1}{2k + 1} \\ &\equiv 2t - 2pt - pt H_n + 2pt \left(H_{p-1} - \frac{H_n}{2} \right) \\ &\equiv 2t - 2pt + 4pt q_p(2) \pmod{p^2} \end{aligned}$$

and hence

$$T_n(pt) \equiv 1 + 2t + 4pt q_p(2) \pmod{p^2}.$$

Therefore, with the helps of Lemma 3.1 and Remark 3.1, we have

$$\begin{aligned}
& T_n(a) - (1 + 2t + 4ptq_p(2)) \\
& \equiv 2 \sum_{k=1}^{\langle a \rangle_p} \binom{m_k + pt - 1}{n} \binom{-1 - pt - m_k}{n} \\
& \equiv 2 \sum_{k=1}^{\langle a \rangle_p} \frac{p(t + \delta_k)}{m_k} = 2 \sum_{j=1}^{\langle a \rangle_p} \frac{pt}{j} + 2 \sum_{\substack{j=1 \\ j > p/2}}^{\langle a \rangle_p} \frac{1}{j} \pmod{p^2},
\end{aligned}$$

where δ_k takes 1 or 0 according as $m_k > p/2$ or not. Below we deal with $a = -1/6, -1/4$.

Clearly,

$$H_{p-k} = H_{p-1} - \sum_{0 < j < k} \frac{1}{p-j} \equiv H_{k-1} \pmod{p}$$

for all $k = 1, \dots, p-1$. Thus, with the help of Lemma 3.3 we have

$$H_{\lfloor 3p/4 \rfloor} \equiv H_{p-1-\lfloor 3p/4 \rfloor} = H_{\lfloor p/4 \rfloor} \equiv -3q_p(2) \pmod{p}$$

and

$$H_{\lfloor 5p/6 \rfloor} \equiv H_{p-1-\lfloor 5p/6 \rfloor} = H_{\lfloor p/6 \rfloor} \equiv -2q_p(2) - \frac{3}{2}q_p(3) \pmod{p}.$$

Case I. $\langle a \rangle_p < n$.

If $a = -1/6$, then $p \equiv 1 \pmod{6}$, $\langle a \rangle_p = (p-1)/6$ and $t = -1/6$. By the above,

$$\begin{aligned}
T_n\left(-\frac{1}{6}\right) & \equiv \frac{2}{3} - \frac{2}{3}pq_p(2) - \frac{p}{3}H_{\lfloor p/6 \rfloor} \\
& \equiv \frac{2}{3} - \frac{2}{3}pq_p(2) - \frac{p}{3}\left(-2q_p(2) - \frac{3}{2}q_p(3)\right) \\
& \equiv \frac{2}{3} + \frac{p}{2}q_p(3) \pmod{p^2}
\end{aligned}$$

and thus

$$\sum_{k=0}^n \frac{\binom{6k}{3k} \binom{3k}{k}}{(2k+1)432^k} = \frac{3}{2}T_n\left(-\frac{1}{6}\right) \equiv 1 + \frac{3}{4}pq_p(3) = \frac{3^p+1}{4} \pmod{p^2}.$$

If $a = -1/4$, then $p \equiv 1 \pmod{4}$, $\langle a \rangle_p = (p-1)/4$ and $t = -1/4$.
By the above,

$$\begin{aligned} T_n \left(-\frac{1}{4} \right) &\equiv \frac{1}{2} - pq_p(2) - \frac{p}{2} H_{\lfloor p/4 \rfloor} \\ &\equiv \frac{1}{2} - pq_p(2) - \frac{p}{2} (-3q_p(2)) \\ &\equiv \frac{1}{2} + \frac{p}{2} q_p(2) \pmod{p^2} \end{aligned}$$

and thus

$$\sum_{k=0}^n \frac{\binom{4k}{2k} \binom{2k}{k}}{(2k+1)64^k} = 2T_n(-1/4) \equiv 1 + pq_p(2) = 2^{p-1} \pmod{p^2}.$$

Case II. $\langle a \rangle_p > n$.

If $a = -1/6$, then $p \equiv 5 \pmod{6}$, $\langle a \rangle_p = (5p-1)/6$ and $t = -5/6$.
By the above,

$$\begin{aligned} T_n \left(-\frac{1}{6} \right) &\equiv -\frac{2}{3} + \frac{2}{3} pq_p(2) + \frac{p}{3} H_{\lfloor 5p/6 \rfloor} \\ &\equiv -\frac{2}{3} + \frac{2}{3} pq_p(2) + \frac{p}{3} \left(-2q_p(2) - \frac{3}{2} q_p(3) \right) \\ &\equiv -\frac{2}{3} - \frac{p}{2} q_p(3) \pmod{p^2} \end{aligned}$$

and hence

$$\sum_{k=0}^n \frac{\binom{6k}{3k} \binom{3k}{k}}{(2k+1)432^k} = \frac{3}{2} T_n \left(-\frac{1}{6} \right) \equiv -1 - \frac{3}{4} pq_p(3) = -\frac{3^p+1}{4} \pmod{p^2}.$$

If $a = -1/4$, then $p \equiv 3 \pmod{4}$, $\langle a \rangle_p = (3p-1)/4$ and $t = -3/4$.
So

$$\begin{aligned} T_n \left(-\frac{1}{4} \right) &\equiv -\frac{1}{2} + pq_p(2) + \frac{p}{2} H_{\lfloor 3p/4 \rfloor} \\ &\equiv -\frac{1}{2} + pq_p(2) + \frac{p}{2} (-3q_p(2)) \\ &\equiv -\frac{1}{2} - \frac{p}{2} q_p(2) \pmod{p^2} \end{aligned}$$

and hence

$$\sum_{k=0}^n \frac{\binom{4k}{2k} \binom{2k}{k}}{(2k+1)64^k} = 2T_n \left(-\frac{1}{4} \right) \equiv -1 - pq_p(2) = -2^{p-1} \pmod{p^2},$$

The proof of Theorem 1.2 is now complete. \square

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